I2205 E – INFO318 E

Graph Theory

Final 2019-2020
Wed. Feb. 26th, am
Duration: 2 hours

Exercise I: Fundamentals (10 minutes - 12 points)

Consider an undirected graph $G = (V, E)$ with nonnegative weights $w(i,j) \geq 0$ on its edges $ij \in E$. Let s be a node in G. Assume you have computed the shortest paths from s, and minimum spanning tree of the graph. Suppose now the weight of every edge in G is increased by 1: $w'(i,j) = w(i,j) + 1$ for every $ij \in E$:

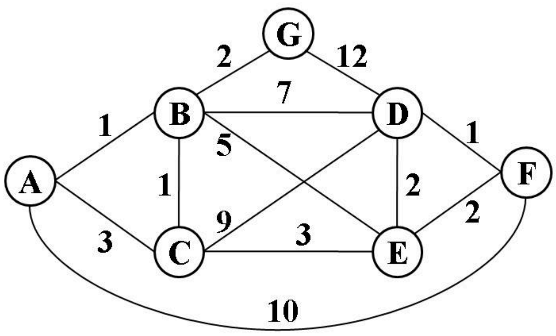
1. Would the minimum spanning tree change? Give an example where it changes or prove that it cannot change.
2. Would the shortest paths change? Give an example where at least one changes or prove that they cannot change.

Solution

1. Yes. If G has n vertices, then any spanning tree of G has n - 1 edges. Therefore, incrementing each edge weight by 1 increases the cost of every spanning tree by a constant, n - 1. So any spanning tree with minimal cost inthe original graph also has minimal cost in the new graph.
2. No. Suppose, for example, that G consists of an edge from s to t of weight 1, and edges from s to a, a to b, and b to t each of weight 0. Then the shortest path is s → a → b → t, with cost 0. But when we increment each edge weight by 1, the shortest path becomes s → t, with cost 2. Again, some people misinterpreted the question, and said the answer is no because the cost of the shortest path changes trivially, even if the set of edges in the path does not change.

Exercise II: Nearest Conference Room (15 minutes - 15 points)

Consider the following undirected, weighted graph, with 7 vertices and 14 edges. The vertices represent the conference rooms in the faculty. An arc ij indicates that there exist a direct path between rooms i and j, while the weights represent length of this path, where the unit is 100 m.



1. Step through Dijkstra's algorithm to calculate the single-source shortest paths from the room A to all the other rooms. Show your steps in a table.
2. Give the obtained shortest paths tree, of root A, showing the distances from A to every other room, in meters.

Solution

A	B	C	D	E	F	G
0	∞	∞	∞	∞	∞	∞
•	1A	3A	∞	∞	10A	∞
•	•	2B	8B	6B	10A	3B
•	•	•	8B	5C	10A	3B
•	•	•	8B	5C	10A	•
•	•	•	7E	•	7E	•
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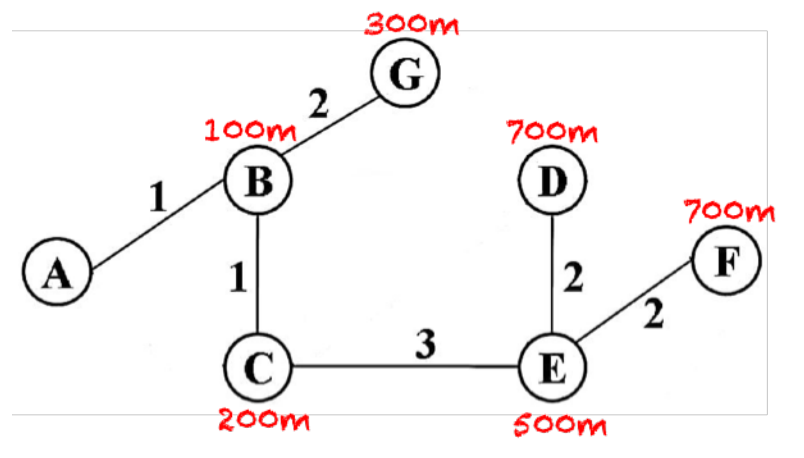


Exercise III: Beirut Rescue (40 minutes - 20 points)

The latest protests in Beirut downtown did not go smoothly and some participants had to be taken to medical emergency treatment at Rafic Hariri University Hospital. In total 150 injured had to get a transfusion of one bag of blood. The hospital had 155 bags in stock. The distribution of blood groups in the supply and amongst the injured in need on blood is shown in the table below (the last part is written as demand).

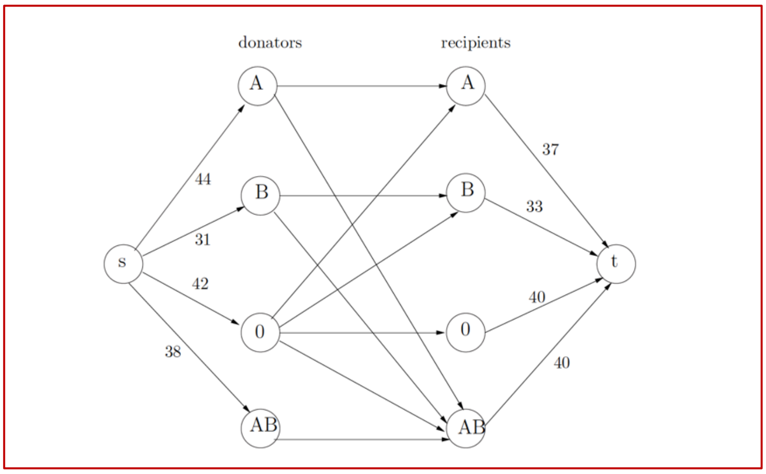
Blood type	A	B	O	AB
Bags in stock	44	31	42	38
Demand	37	33	40	40

- *Type A patients can only receive blood of type A or type O.*
- *Type B patients can receive only type B or type O.*
- *Type O patients can receive only type O.*
- *Type AB patients can receive any of the four types.*

1. Represent the blood distribution problem as a flow network N. Draw N showing the vertices, arcs and their capacities.
2. Say what flow value you are looking for and how to interpret it as a proper assignment of blood bags.
3. Solve the problem applying the adequate algorithm, in order to satisfy the maximum number of blood transfer demands.
4. Provide the optimal assignment of blood bags, that is found via your flow algorithm above.

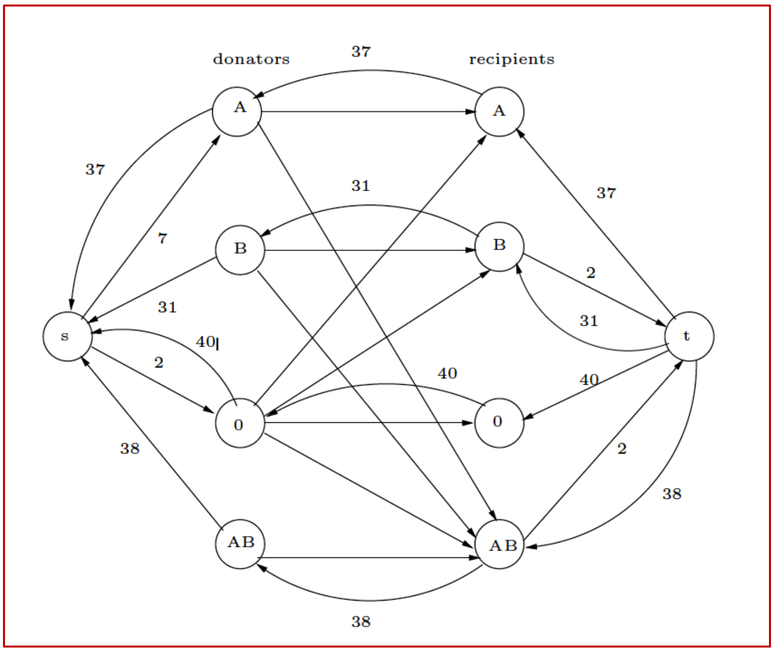
Solution

1. The proposed flow network N for the blood distribution problem. All capacities on the arcs in the middle are infinite. We can use this network to model the problem of checking whether it is possible to give all participants a blood transfusion that is compatible with the restrictions above. The capacities of the arcs from s indicate how much blood we have in stock of each type and the capacities on the arcs into t show the demand for each of the four types of recipients. The arcs in the middle model exactly which types of blood can be given to a patient with a given blood type.



2. We want the flow to model the assignment of blood bags to participants so we are looking for an integer flow of value 150. The flow on an arc (i → j) in the middle will then tell us how many bags of type i we will use to satisfy (a part of) the demand for participants with blood type j. We can replace the infinite capacities on arcs (i → j) by the capacity of the arc from s to t (which equals the demand for that type).
3. We start by sending flow along the following 4 paths which are disjoint apart from s, t:
   - sAAt → 37 units
   - sBBt → 31 units
   - s0Ot → 40 units
   - sABABt → 38 units,

   This gives us a flow of value 146. The residual network now looks as follows (infinite capacities not explicitly written)



   Now we identify the following augmenting paths which again share only s, t: s0Bt capacity 2 and sAABt capacity 2. Augmenting along both of these gives us a maximum flow of values 150 (all arcs into t are filled).
4. The optimal assignment that can be read off from the final flow values are
   - A → A: 37 bags,
   - A → AB: 2 bags
   - B → B: 31 bags
   - 0 → 0: 40 bags,
   - 0 → B: 2 bags
   - AB → AB: 38 bags..

Exercise IV: MEA Crisis (40 minutes - 23 points)

Middle East Airlines (MEA) connects Beirut ***BEY*** and 7 cities each in a region: North America (New York ***JFK***), South America (Sao Paulo ***SAO***), Europe (Paris ***CDG***), Africa (Abidjan ***ABJ***), Middle East (Istanbul ***IST***), The Gulf (Dubai ***DXB***), and Australia (Sydney ***SYD***).

Network

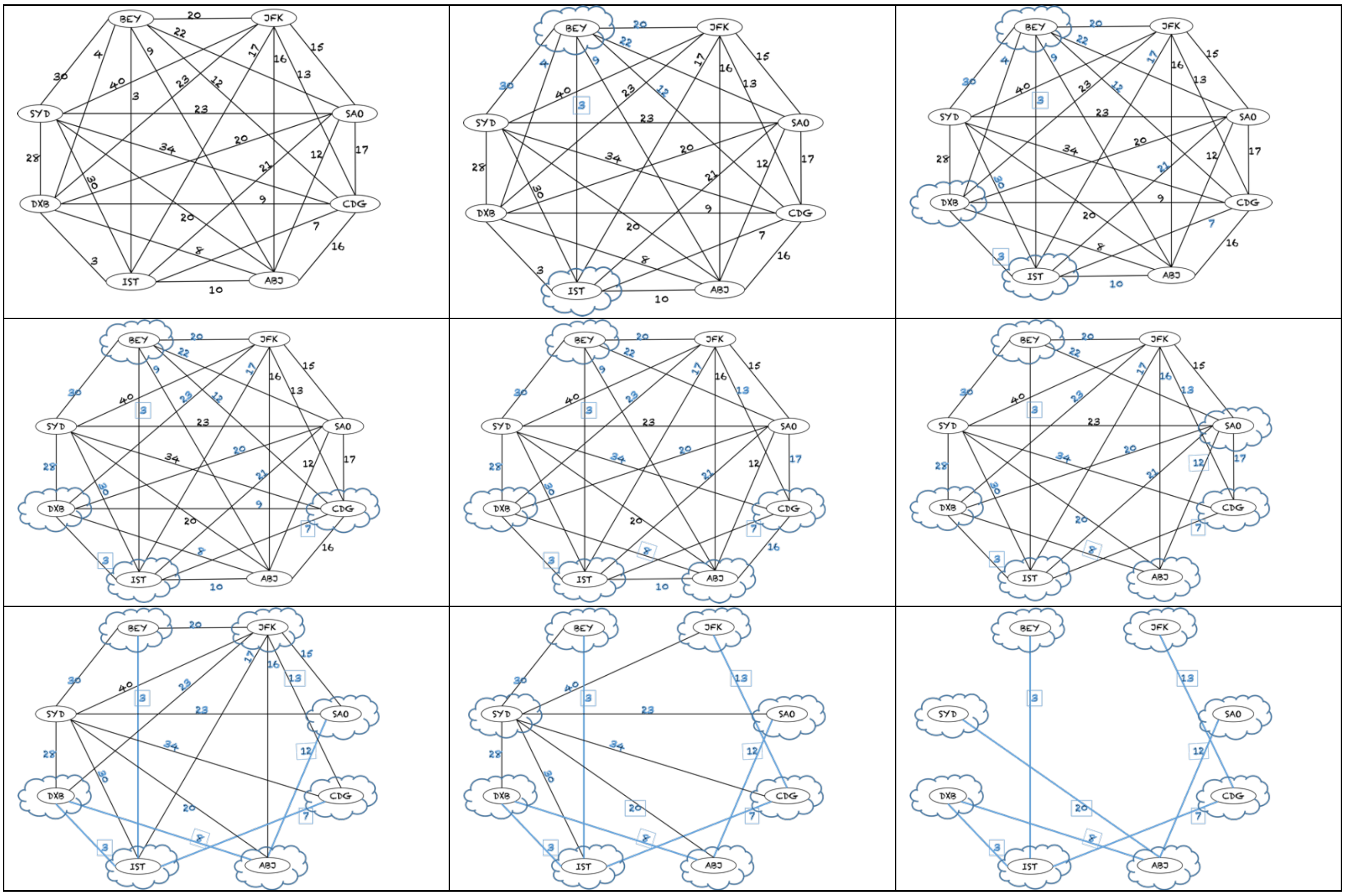
The fares, in hundreds of dollars, for direct flights between the corresponding airports are given in the following table, note that the matrix is symmetric:

	BEY	JFK	SAO	CDG	ABJ	IST	DXB	SYD
BEY		20	22	12	9	3	4	30
JFK			15	13	16	17	23	40
SAO				17	12	21	20	23
CDG					16	7	9	34
ABJ						10	8	20
IST							3	30
DXB								28
SYD								

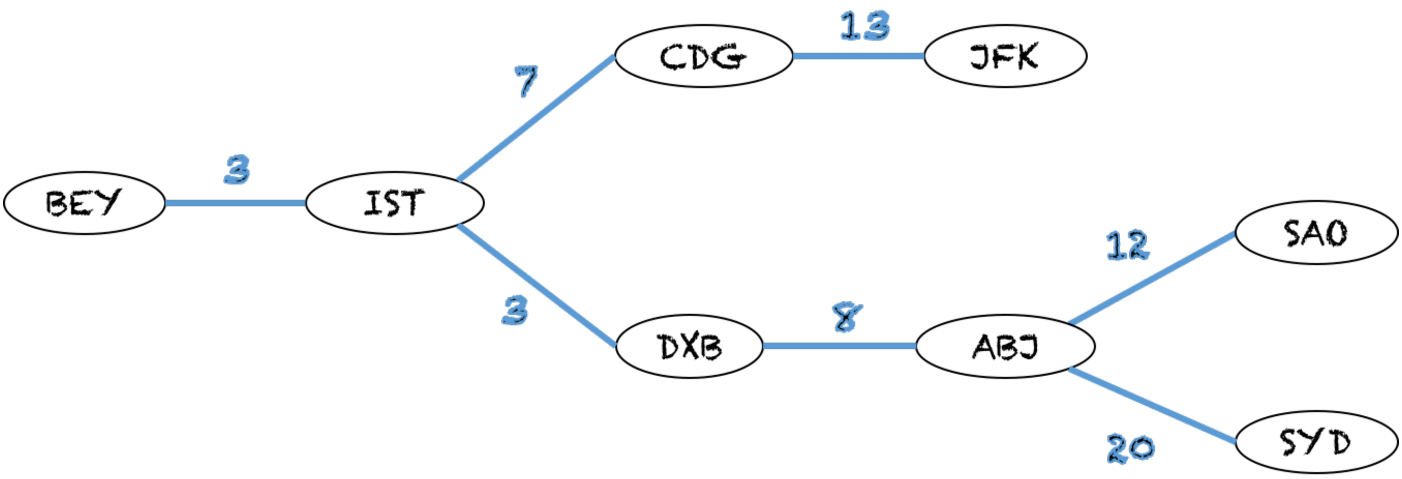
1. Consider the flights graph, where the vertices represent the 8 airports and the edges represent the direct flights, weight ed with the costs. Is it a complete graph? Why?
2. MEA is facing financial problems due to the economic crisis touching Lebanon. The company decides to remove some direct flights from its network, while keeping it connected with the minimum number of edges and minimum total cost. Apply a suitable algorithm which indicates the flights to be kept. What is the type of the obtained network?
3. Suppose now, there is a constraint to the MEA decision, by which it wants to keep the direct flight between Beirut and Paris due to political reasons. What approach would you take in order to satisfy the request, while keeping minimum possible flights and costs? Explain. You are not to apply your approach, just explain.

Solution

1. First figure in table below. Yes complete. There is an arc from every vertex to every other as per the adjacency matrix.
2. Kruskal starting from BEY (any other start would lead to the same tree)



We obtain the following minimal spanning tree



3. Modify Prim's algorithm application by starting with the connected forest = {vertices BEY and PDG with their connecting arc} continue normally.