Lebanese University
Faculty of Sciences 1

I2205 E – INFO318 E

Graph Theory

Partial 2019-2020
Tue. Dec. 13th, pm
Duration: 1 hour

Exercise I: Fundamentals (7 points)

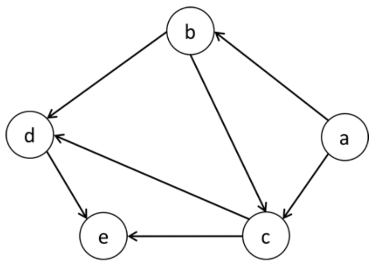
Part A

Consider the directed graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{(a, b),\ (a, c),\ (b, c),\ (b, d),\ (c, d),$ $(c, e),\ (d, e)\}$.

1. Find the adjacency matrix **and** the adjacency list of the graph G. Deduce the out-degrees and the in-degrees of the vertices.
2. Is this graph bipartite? Why or why not?
3. Is this graph Eulerian? Deduce a necessary condition for a directed graph to be Eulerian.

solution

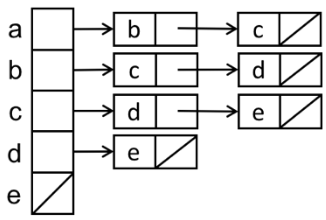
1.



Graph G

	a	b	c	d	e
a	0	1	1	0	0
b	0	0	1	1	0
c	0	0	0	1	1
d	0	0	0	0	1
e	0	0	0	0	0

Adjacency matrix



Adjacency list

vertex	in-degree	out-degree
a	0	2
b	1	2
c	2	2
d	2	1
e	2	0

Degrees

2. G is not bipartite because it contains odd length cycles (direction ignored).
3. G is not eulerian. Vertex e has two in edges and zero out edge. It ca not be visited.
A directed graph has an Euler circuit if and only if for all v in G, indeg(v) = outdeg(v).
readings:
http://www.bowdoin.edu/~ltoma/teaching/cs231/fall09/Homeworks/old/H9-sol%203.pdf
https://math.stackexchange.com/questions/1871065/euler-path-for-directed-graph

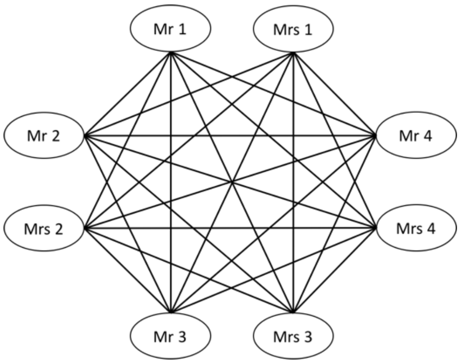
Part B

Four couples met in a conference, where every one has to shakehand with the others, except for his spouse.

1. Represent the given problem by a graph where the vertices and the edges be determined.
2. Is the obtained graph complete? Eulerian? Explain.
3. Is the obtained graph Hamiltonian? If yes give a Hamiltonian cycle. What does such a cycle mean?

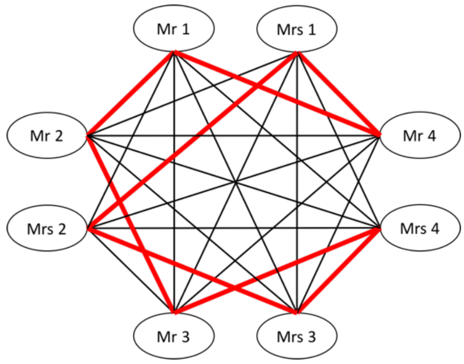
solution

1.



Graph G

vertex = person
edge = shakehand
$\boldsymbol{G = (V, E)}$



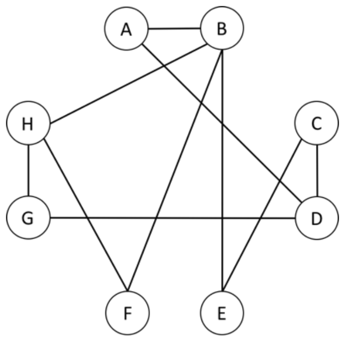
Hamiltonian cycle in G

2. G is regular but not complete. No egdes between husbands and wifes. G is 6-regular. All vertices are of degree 6 (even). So G is Eulerian.

3. G is Hamiltonian. Hamiltonian cycle= Mr1, Mr4, Mrs1, Mrs2, Mrs3, Mrs4, Mr3, Mr2, Mr1. Such a cycle means that they can shake hands simultaneously (each one handshakes his two neighbors, using a hand for each).

Exercise II: Graph coloring (6 points)

Consider the following graph $G = (V, E)$:



1. Maximize and minimize the chromatic number, χ(G), of G.
2. Applying the algorithm of your choice, propose a good coloring for the vertices of G.
3. Propose a good coloring for the edges of G.

solution

1. Minimization: there exists a complet subgraph of size 3 $\Rightarrow \boldsymbol{\chi(G) \geq 3}$. Also because there exist an odd cycle. Maximization: maximum degree is 4. $\boldsymbol{\deg(B) = 4 \Rightarrow \chi(G) \leq 5}$. So, $\boldsymbol{3 \leq \chi(G) \leq 5}$.

2. Vertex coloring algorithm of Welch and Powell.

vertex coloring

degree	4	3	3	2	2	2	2	2
vertex	B	D	H	A	C	E	F	G

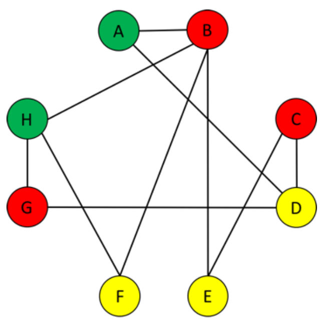
B←color 1, also C and G

D← color2, also E and F

H←color 3 also A

degree	4	3	3	2	2	2	2	2
vertex	B	D	H	A	C	E	F	G
color	1	2	3	3	1	2	2	1

3 colors, therefor optimal coloring



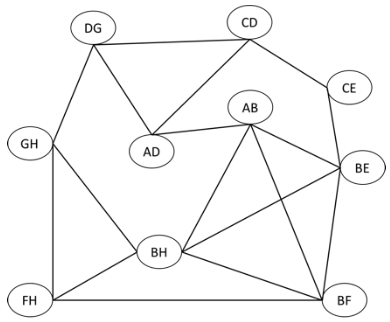
3. Finding the minimum edge coloring of a graph G is equivalent to finding the minimum vertex coloring of its line graph L(G). A line graph L(G) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common.

Minimization: there exists a complet subgraph of size 4 $\Rightarrow \chi(L(G)) \geq 4$. Maximization: maximum degree is 5. $\deg(BH) = 5 \Rightarrow \chi(L(G)) \leq 6$. So, $4 \leq \chi(L(G)) \leq 6$.

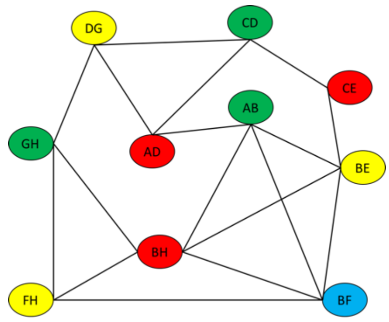
edge coloring of G = vertex coloring of line graph L(G)

degree	5	4	4	4	3	3	3	3	3	2
vertex	BH	AB	BE	BF	CD	DG	GH	FH	AD	CE
color	1	2	3	4	2	3	2	3	1	1

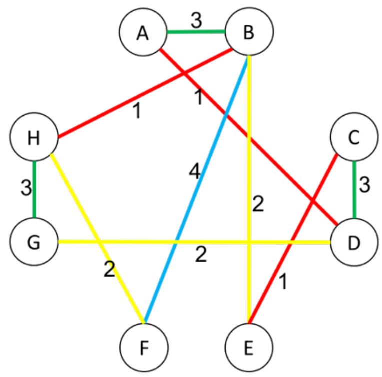
4 colors, therefor optimal coloring



Line graph L(G)



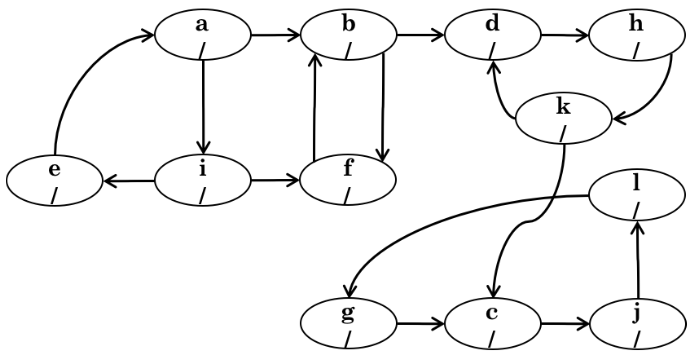
Vertex coloring of L(G)



Edge coloring of G

Exercise III: Graph Traversal (7 points)

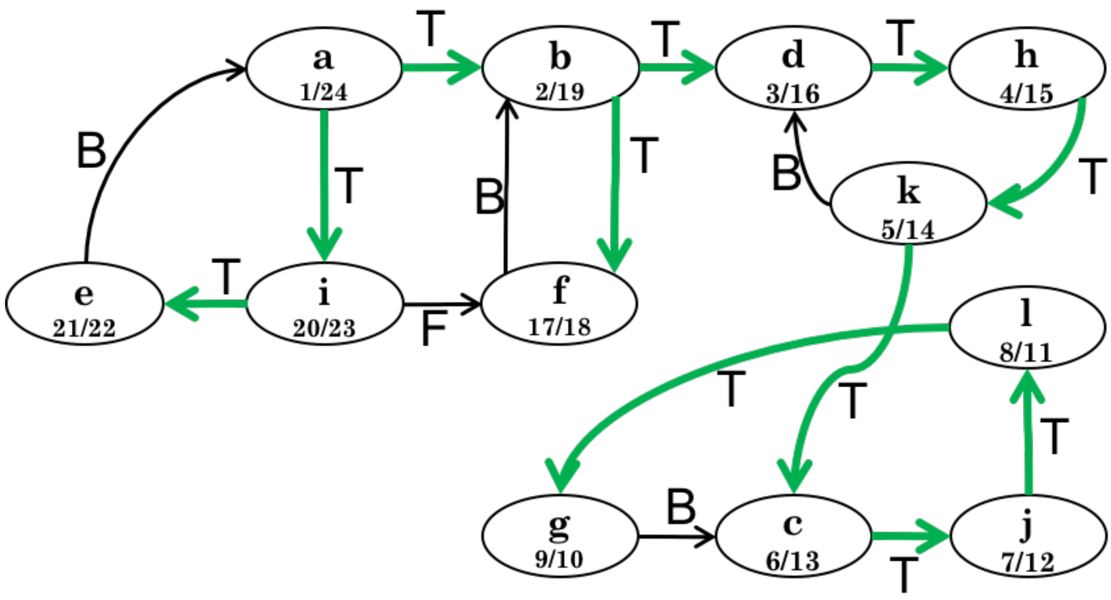
Consider the following graph $G = (V, E)$:



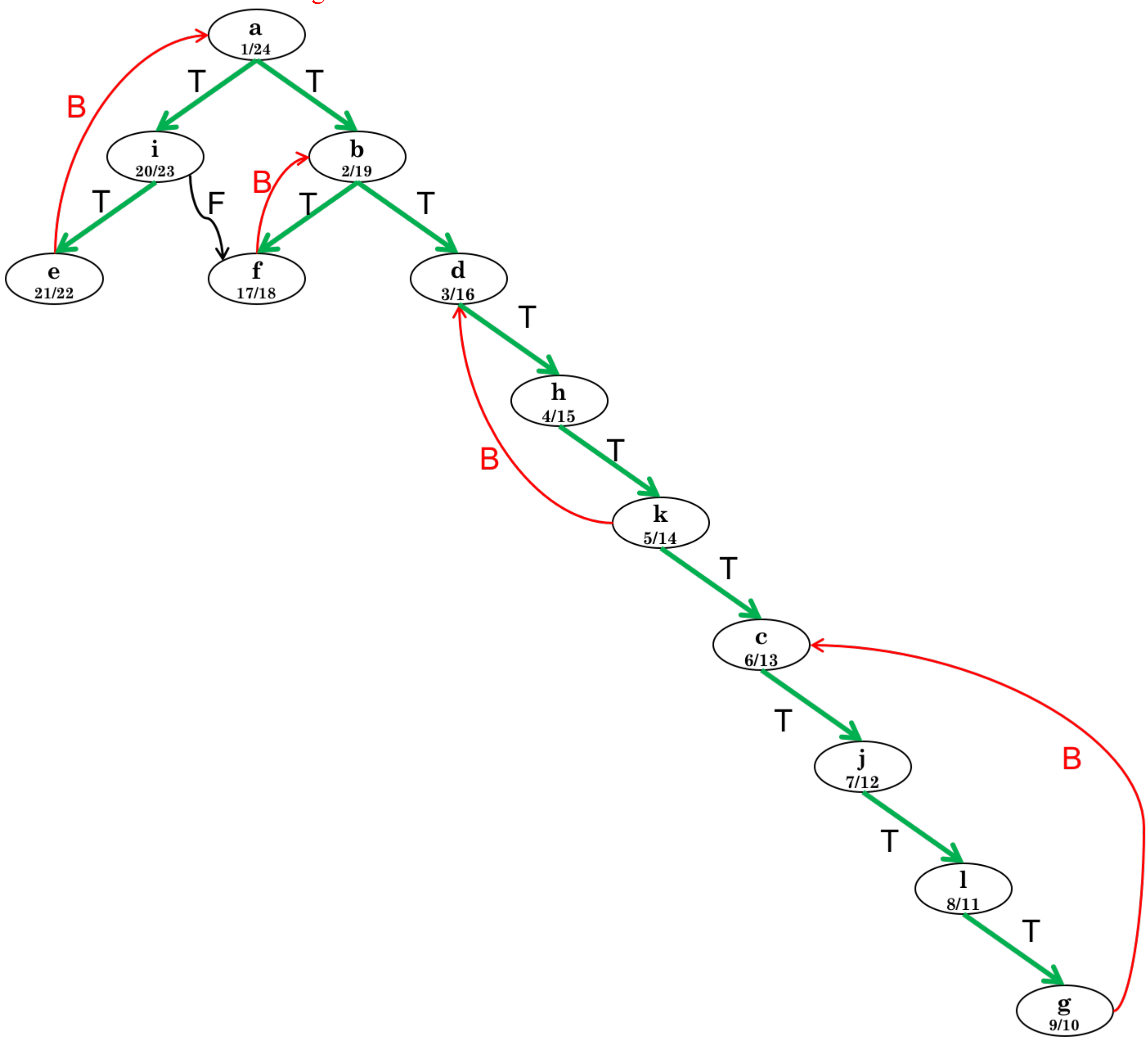
1. Apply the DFS algorithm on the given graph starting at vertex a. If there is a decision to be taken between several vertices, choose the letter closest to the beginning of the alphabet first. Indicate the discovering time (d), finishing time (f), the tree edge (T), forward edge (F), backward edge (B), and the cross edge (C).
2. Give the DFS tree.
3. Does this graph have bridges? If yes, give them.
4. Determine the circuits in G and give the reduced graph G_R.

solution

1. DFS



1. DFS tree is shown in the figure below.



3. G contains 2 bridges: (b,d) and (k,c).
4. Circuits in G are found thankx to backward edges: (a,i,e); (b,f);(d,h,k);(c,j,l,g).

5. In the reduced graph GR, shown in blue below, we replace each circuit by one vertex. G contains 4 strobly connected components c1,c2,c3,c4, where:
$c1 = (\mathrm{E1}, \mathrm{V1});\ with\ E1 = \{a, e, i\}\ and\ V1 = \{(a, i), (i, e), (e, a)\}$,
$c2 = (\mathrm{E2}, \mathrm{V2});\ with\ E2 = \{b, \mathrm{f}\}\ and\ V2 = \{(b, f), (f, b)\}$,
$c3 = (\mathrm{E3}, \mathrm{V3});\ with\ E3 = \{d, h, k\}\ and\ V3 = \{(d, h), (h, k), (k, d)\}$,
$c4 = (\mathrm{E4}, \mathrm{V4});\ with\ E4 = \{c, j, j, g\}\ and\ V4 = \{(c, j), (j, l), (l, g), (g, c)\}$, and
$\mathrm{G}_R = (\mathrm{E}_R, \mathrm{V}_R);\ with\ \mathrm{E}_R = \{c1, c2, c3, c4\}\ and\ \mathrm{V}_R = \{(c1, c2), (c2, c3), (c3, c4)\}$.